Fluctuation-dissipation relation in an Ising model without detailed balance

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We consider the modified Ising model introduced by de Oliveira, Mendes, and Santos [J. Phys. A **26**, 2317 (1993)], where the temperature depends locally on the spin configuration and detailed balance and local equilibrium are not obeyed. We derive a relation between the linear response function and correlation functions that generalizes the fluctuation-dissipation theorem. In the stationary states of the model, which are the counterparts of the Ising equilibrium states, the fluctuation-dissipation theorem breaks down due to the lack of time reversal invariance. In the nonstationary phase-ordering kinetics, the parametric plot of the integrated response function $\chi(t, t_w)$ vs the autocorrelation function is different from that of the kinetic Ising model. However, splitting $\chi(t, t_w)$ into a stationary and an aging term $\chi(t, t_w) = \chi_{st}(t - t_w) + \chi_{ag}(t, t_w)$, we find $\chi_{ag}(t, t_w) \sim t_w^{-a_\chi} f(t/t_w)$, and a numerical value of a_χ consistent with $a_\chi = \frac{1}{4}$, as in the kinetic Ising model.

I. INTRODUCTION

In equilibrium systems the integrated response function $\chi(t, t_w)$ and the autocorrelation function $C(t, t_w)$ depend only on the two-time difference $\tau = t - t_w$ and are related by the fluctuation-dissipation theorem (FDT) [1]

$$\chi(\tau) = \hat{\chi}(C), \tag{1}$$

where

$$\hat{\chi}(C) = \beta[C(0) - C(\tau)] \tag{2}$$

and $\beta = -d\hat{\chi}(C)/dC$ is the inverse equilibrium temperature.

In recent times, many studies have considered the possibility to relate response and correlation functions in nonequilibrium systems. In aging systems, such as glassy materials and coarsening systems, relaxation properties depend both on t_w and t and FDT breaks down. In this context, guided by the solution of mean field spin glass models, Cugliandolo and Kurchan [2] proposed that, despite the explicit two-time dependence of χ and C, a relation analogous to Eq. (1), namely,

$$\chi(t, t_w) = \hat{\chi}(C), \tag{3}$$

may still hold for a large class of systems in the large t_w limit. The functional form of $\hat{\chi}(C)$, however, is different from the equilibrium one (2) and system dependent. In particular, it was shown [3] that $\hat{\chi}(C)$ is related to basic properties of the equilibrium states; because of that, aging systems can be classified [4] into few classes according to the shape of $\hat{\chi}(C)$. Phase ordering systems are characterized by a broken line shape of $\hat{\chi}(C)$. More precisely, for C larger than the Edwards-Anderson order parameter q_{EA}, FDT (2) still holds. For $C \le q_{EA}$ one has a horizontal line, namely, a constant

integrated response function. In analogy to equilibrium systems, the quantity $-d\hat{\chi}(C)/dC$ can be interpreted [5] as an *effective* inverse temperature $\beta_{\text{eff}}(C)$ of the nonequilibrium state. In coarsening systems this quantity takes two values, the temperature of the reservoir $\beta_{\text{eff}}(C) = \beta$, in the region $C > q_{EA}$, and $\beta_{\text{eff}}(C) = 0$, for $C \le q_{EA}$. The feature $\beta_{\text{eff}}(C) = \beta$ in the region of the largest values of $C(t, t_w)$ is quite general in systems where local equilibrium [6] is obeyed. Local equilibrium, in fact, implies that on short time scales FDT holds. Since small time separations correspond to the largest values of $C(t, t_w)$, in this regime one has $\beta_{\text{eff}}(C) = \beta$. The situation is different in systems where local equilibrium is not obeyed. In this case, one does not expect to observe FDT even in the short time-scale regime and the definition of a thermodynamic temperature from $\hat{\chi}(C)$ may be incorrect.

In this paper we study the fluctuation-dissipation relation in a two-dimensional Ising model without detailed balance (IWDB), originally introduced in Ref. [7], for which local equilibrium does not hold. This spin system is analogous to the kinetic Ising model (KIM), but the temperature entering the transition rates depends on space and time through the system configuration. This model is known to behave much like the KIM. In the phase diagram, a disordered high-temperature phase and a low-temperature phase with ergodicity breaking are separated by a critical line. The phase transition is characterized by the same critical exponents [7] of the Ising model. After a quench into the low-temperature phase, the nonstationary dynamics is analogous [8] to phase ordering in the KIM.

Despite these strong similarities, lack of detailed balance makes the IWDB, in principle, much different from the KIM and gives rise to interesting features that can be enlightened by the fluctuation-dissipation relation. In the KIM, stationary states are equilibrium states characterized by time reversal invariance (TRI). Instead, in the stationary states of the IWDB, which are the counterparts of the KIM equilibrium states, TRI is violated and FDT (2) breaks down. The relation between $\chi(t, t_w)$ and $C(t, t_w)$ may not be meaningful, as in systems with detailed balance, and $-d\hat{\chi}(C)/dC$ cannot be

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straightforwardly interpreted as a thermodynamic temperature. Interestingly, however, we can derive fluctuation-dissipation relations that generalize to the present model what is known in systems where detailed balance holds. In doing that, we uncover that the response function is not naturally related to the spin autocorrelation function $C(t,t_w)$ $=\langle\sigma_i(t)\sigma_i(t_w)\rangle$, but rather to the quantity $A(t,t_w)$ $=\langle\sigma_i(t)\sigma_i(t_w)\beta_i(t_w)\rangle$, $\beta_i(t_w)$ being the space-time-dependent inverse temperature. In stationary states, the fluctuation-dissipation relation reads

$$\chi(\tau) = A(0) - A(\tau) - \Delta(\tau),\tag{4}$$

where $\Delta(\tau)$ is a quantity related to the lack of TRI, analogous to the *asymmetry* [9] in systems with detailed balance out of equilibrium. In the special case when the temperature is constant, $\beta_i(t) = \beta$, the model reduces to the KIM. Since now TRI is recovered, one has $\Delta(\tau) = 0$ and then

$$\chi(\tau) = A(0) - A(\tau).\tag{5}$$

Recalling that in this case $A(\tau) = \beta C(\tau)$ one recognizes the FDT, Eqs. (1) and (2). In the case with nonconstant temperature, since $A(t,t_w) \neq \beta C(t,t_w)$ both in stationary and nonstationary states, the relation between $\chi(t,t_w)$ and $C(t,t_w)$ remains unclear.

In the phase-ordering process following a temperature quench, two-time quantities can be split into a stationary and an aging term. In particular, for the integrated response function one has $\chi(t,t_w) = \chi_{st}(t-t_w) + \chi_{ag}(t,t_w)$, where the aging part obeys the scaling form

$$\chi_{ag}(t,t_w) = t_w^{-a_\chi} f(y),\tag{6}$$

with $y = t/t_w$, as generally expected [11] in phase-ordering systems. When detailed balance holds, the exponent a_χ is uniquely determined by the space dimensionality d and the dynamic exponent z. Since in the IWDB the value of z is the same [8] of the KIM, one expects the same value of a_χ in the two models. In fact, in $d = 2$ we find results consistent with $a_\chi = \frac{1}{4}$, as in the KIM. This result complements those of Refs. [7,8], where it was found that the *equilibrium* critical exponents and the *nonequilibrium* persistence exponent of the IWDB were the same, within statistical errors, of those of the Ising model. The numeric results of this paper, therefore, strengthen the idea that the two models belong to the same nonequilibrium universality class.

Despite this, the parametric plot of $\hat\chi(C)$ is different from that of the KIM. Interestingly, the shape of this function is similar to that found [10] in a soluble model of sheared binary systems where detailed balance is also violated, but local equilibrium still holds. In particular, the extrapolation of the numerical data to the large t_w limit is consistent with the horizontal line typical of phase ordering in the region of small C of the plot.

This paper is organized as follows: In Sec. II, we introduce the model. In Sec. III, we derive a relation between the response function and correlation functions that generalizes the FDT to the present model. This relation allows one to compute, numerically, the response function without applying a perturbation and to discuss the fluctuation-dissipation

relation in Sec. IV. In particular, the stationary states at high and low temperature are discussed in Secs. IV A 1 and IV A 2, while the aging dynamics following a quench is considered in Sec. IV B. A summary and the conclusions are drawn in Sec. V.

II. THE MODEL

We consider the Ising model defined by the Hamiltonian

$$H[\sigma] = -J\sum_{\langle ij\rangle}\sigma_i\sigma_j = -\sum_i\sigma_i H_i[\sigma],\tag{7}$$

where $\sigma_i = \pm 1$ is a spin variable on a d-dimensional lattice and $\langle ij\rangle$ denotes nearest neighbors i,j sites. $H_i[\sigma] = J\sum_{j_i}\sigma_{j_i}$, where j_i runs over the nearest neighbors of i, is the local field.

A dynamics is introduced by randomly choosing a single spin on site i and updating it in an elementary time step with a transition rate $w([\sigma]\to[\sigma'])$. Here, $[\sigma]$ and $[\sigma']$ are the spin configurations before and after the move, which differ only by the value of σ_i. In the IWDB, transition rates are generic but their ratio must fulfill the condition

$$\frac{w([\sigma]\to[\sigma'])}{w([\sigma']\to[\sigma])} = \exp\left\{-\sum_i \beta_i[\sigma](\sigma_i H_i[\sigma] - \sigma_i' H_i[\sigma'])\right\}.\tag{8}$$

With a constant $\beta_i[\sigma] = \beta$, one recovers the kinetic Ising model (KIM) in contact with a reservoir at the temperature $T = \beta^{-1}$. In this case, Eq. (8) is the detailed balance condition with respect to the Hamiltonian (7). In fact, one has

$$-\sum_i \beta(\sigma_i H_i[\sigma] - \sigma_i' H_i[\sigma']) = \beta(H[\sigma] - H[\sigma'])\tag{9}$$

and, hence,

$$\frac{w([\sigma]\to[\sigma'])}{w([\sigma']\to[\sigma])} = \frac{\exp\{-\beta H[\sigma']\}}{\exp\{-\beta H[\sigma]\}} = \frac{P_{eq}[\sigma']}{P_{eq}[\sigma]},\tag{10}$$

where

$$P_{eq}[\sigma] \propto \exp\{-\beta H[\sigma]\}\tag{11}$$

is the canonical equilibrium probability. Detailed balance implies that stationary states of the model are also equilibrium states with measure (11), which are characterized by TRI. For a generic two-time quantity $F(t,t_w)$, TRI implies $F(t,t_w) = F(-t,-t_w)$. If TTI is also obeyed, namely, $F(t,t_w) = F(t-t_w)$, by shifting time by an amount t_w, one also has $F(t,t_w) = F(|t-t_w|)$.

In this paper, we consider the case when $\beta_i[\sigma]$ is not constant but depends on the configuration through the local field $H_i[\sigma]$, $\beta_i[\sigma] = \beta(H_i[\sigma])$. Physically, one can imagine a system in contact with reservoirs at different temperatures each of which couples to the spins σ_i with the same local field $H_i[\sigma]$. Note that flipping σ_i does not change $H_i[\sigma]$; hence, $\beta_i[\sigma] = \beta_i[\sigma']$. However, this is true only for the site i where the flip occurs, while, in general, $\beta_j[\sigma] \neq \beta_j[\sigma']$ for $j \neq i$. The very basic feature that makes this model different

from the KIM is the fact that its transition rates do not obey detailed balance. This is expected on physical grounds, since different local temperatures in the system determine heat fluxes that break TRI and, hence, detailed balance. Mathematically, this happens because the argument of the exponential in Eq. (8) cannot be written as a difference $\mathcal{H}[\sigma'] - \mathcal{H}[\sigma]$, $\mathcal{H}$ being a generic function, as in Eq. (9), due to the factor $\beta_i[\sigma]$. Indeed, the term $\Sigma_i \beta_i[\sigma] \sigma_i H_i[\sigma]$ is a function $\mathcal{H}[\sigma]$ of the configuration $[\sigma]$ but the term $\Sigma_i \beta_i[\sigma] \sigma'_i H_i[\sigma'] \neq \mathcal{H}[\sigma']$ because it depends on both the configurations $[\sigma]$ and $[\sigma']$. Because detailed balance is not obeyed, the stationary states of the model [8] are not equilibrium states and, in principle, TRI is not expected.

As discussed in [7,8,13], the present model contains, as special cases corresponding to particular choices of $\beta(|H_i|)$, the Voter, majority Voter, and noisy Voter model, besides, clearly, the KIM.

III. FLUCTUATION-DISSIPATION RELATIONS

In this section we derive a relation between the response function and particular correlation functions that generalize the result of Ref. [14] to the case of a nonconstant $\beta_i[\sigma]$. The derivation closely follows that of Ref. [14] to which we refer for further details.

Let us consider a perturbing magnetic field switched on the jth site in the time interval $[t', t' + \Delta t]$,

$$h_i(t) = h\delta_{i,j}\theta(t - t')\theta(t' + \Delta t - t), \tag{12}$$

where θ is the Heaviside step function. The Hamiltonian (7) is changed to

$$H[\sigma] = -J\sum_{\langle ij \rangle} \sigma_i \sigma_j - \sum_i h_i(t)\sigma_i = -\sum_i \sigma_i H_i^h[\sigma], \tag{13}$$

where $H_i^h[\sigma] = J\sum_{j_i}\sigma_{j_i} + h_i(t)$. In the limit of vanishing h, the effect of the perturbation (12) on the spin on site i at the time $t > t'$ is given by the linear response function [15,16]

$$R_{i,j}(t,t') = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left. \frac{\partial \langle \sigma_i(t) \rangle}{\partial h_j(t')} \right|_{h=0}, \tag{14}$$

where here and in the following $\langle \cdots \rangle$ means ensemble averages, namely, taken over different initial conditions and thermal histories. Introducing the probability $p([\sigma], t)$ to find the system in the configuration $[\sigma]$ at time t, and the conditional probability $p([\sigma], t | [\sigma'], t')$ to find the configuration $[\sigma]$ at time t given that the system was in the configuration $[\sigma']$ at t', the right-hand side of Eq. (14) can be written as

$$\left. \frac{\partial \langle \sigma_i(t) \rangle}{\partial h_j(t')} \right|_{h=0} = \sum_{[\sigma],[\sigma'],[\sigma'']} \sigma_i p([\sigma], t | [\sigma'], t' + \Delta t)$$

$$\times \left. \frac{\partial p^h([\sigma'], t' + \Delta t | [\sigma''], t')}{\partial h_j} \right|_{h=0} p([\sigma''], t'). \tag{15}$$

Here p and p^h refer to the conditional probabilities of the unperturbed and perturbed system, respectively. Let us concentrate on the factor containing p^h. The conditional probability for Δt sufficiently small is given by

$$p^h([\sigma'], t' + \Delta t | [\sigma''], t')$$

$$= \delta_{[\sigma'],[\sigma'']} + w^h([\sigma''] \to [\sigma'])\Delta t + O(\Delta t^2), \tag{16}$$

where we have used the boundary condition

$$p([\sigma'], t | [\sigma''], t) = \delta_{[\sigma'],[\sigma'']}.$$

Furthermore, also the perturbed transition rates w^h must verify the condition (8), namely,

$$\frac{w^h([\sigma'] \to [\sigma''])}{w^h([\sigma''] \to [\sigma'])}$$

$$= \exp\left\{ -\sum_i \beta_i[\sigma'](\sigma'_i H_i^h[\sigma'] - \sigma''_i H_i^h[\sigma'']) \right\}. \tag{17}$$

Expanding the perturbed transition rates in powers of h, one finds that the following form:

$$w^h([\sigma'] \to [\sigma''])$$

$$= w([\sigma'] \to [\sigma''])\left\{ 1 - \frac{1}{2}\beta_j[\sigma'](\sigma'_j h_j - \sigma''_j h_j) \right\}, \tag{18}$$

where $w([\sigma'] \to [\sigma''])$ are generic unperturbed transition rates obeying (8), is compatible to first order in h with the condition (17).

Using Eqs. (16) and (18), following Ref. [14], the response function can be written as the sum of two contributions

$$R_{i,j}(t,t') = \lim_{\Delta t \to 0} [D_{i,j}(t,t',\Delta t) + \bar{D}_{i,j}(t,t',\Delta t)], \tag{19}$$

where

$$D_{i,j}(t,t',\Delta t)$$

$$= \frac{1}{2} \sum_{[\sigma],[\sigma']} \sigma_i p([\sigma], t | [\sigma'], t' + \Delta t)$$

$$\times \sum_{[\sigma''] \neq [\sigma']} w([\sigma'] \to [\sigma''])\beta_j[\sigma'](\sigma'_j - \sigma''_j)p([\sigma'], t') \tag{20}$$

and

$$\bar{D}_{i,j}(t,t',\Delta t)$$

$$= \frac{1}{2} \sum_{[\sigma],[\sigma'],[\sigma''] \neq [\sigma']} \sigma_i p([\sigma], t | [\sigma'], t' + \Delta t)(\sigma'_j - \sigma''_j)$$

$$\times \beta_j[\sigma']w([\sigma''] \to [\sigma'])p([\sigma''], t'). \tag{21}$$

Using the time translational invariance (TTI) of the conditional probability $p([\sigma], t | [\sigma'], t' + \Delta t) = p([\sigma], t - \Delta t | [\sigma'], t')$, one can write $D_{i,j}(t, t', \Delta t)$ in the form of a correlation function

$$D_{i,j}(t,t',\Delta t) = -\tfrac{1}{2}\langle \sigma_i(t - \Delta t)B_j(t') \rangle, \tag{22}$$

where

$$B_j = -\sum_{[\sigma'']} (\sigma_j - \sigma_j'')\beta_j[\sigma]w([\sigma] \to [\sigma'']). \qquad (23)$$

Using Eq. (16), $\bar{D}_{i,j}(t,t',\Delta t)$ can be written as

$$\bar{D}_{i,j}(t,t',\Delta t) = \frac{1}{2}\frac{\Delta A_{i,j}(t,t')}{\Delta t}, \qquad (24)$$

where

$$\Delta A_{i,j}(t,t') = \langle \beta_j(t')\sigma_i(t)[\sigma_j(t'+\Delta t) - \sigma_j(t')]\rangle. \qquad (25)$$

Therefore, putting together Eqs. (22) and (24) and taking the limit $\Delta t \to 0$, we obtain

$$R_{i,j}(t,t') = \frac{1}{2}\frac{\partial A_{i,j}(t,t')}{\partial t'} - \frac{1}{2}\langle \sigma_i(t)B_j(t')\rangle, \qquad (26)$$

where

$$A_{i,j}(t,t') = \langle \beta_j(t')\sigma_i(t)\sigma_j(t')\rangle. \qquad (27)$$

In the following, we will be interested in the integrated response function:

$$\chi_{i,j}(t,t_w) = \int_{t_w}^{t} R_{i,j}(t,t')dt', \qquad (28)$$

which corresponds to the application of a perturbing field between the times t_w and t. This quantity is easier to measure because switching on the perturbation for a finite time increases the signal to noise ratio. From Eq. (26), we have

$$\chi_{i,j}(t,t_w) = \frac{1}{2}[A_{i,j}(t,t) - A_{i,j}(t,t_w)] - \frac{1}{2}\int_{t_w}^{t}\langle \sigma_i(t)B_j(t')\rangle. \qquad (29)$$

Equations (26) and (29) are the principal results of this section. They are relations between the response function and correlation functions of the unperturbed kinetics, which generalize the FDT. These relations hold both in stationary and nonstationary states and do not depend on the choice of the unperturbed transition rates, provided the condition (8) is obeyed.

From Eq. (29), the integrated autoresponse function $\chi(t,t_w) = \chi_{i,i}(t,t_w)$, which does not depend on i due to space translation invariance, is given by

$$\chi(t,t_w) = \frac{1}{2}[A(t,t) - A(t,t_w)] - \frac{1}{2}\int_{t_w}^{t}\langle \sigma_i(t)B_i(t')\rangle, \qquad (30)$$

where $A(t,t_w) = A_{i,i}(t,t_w)$. We will use Eq. (30) for numerical computations in Sec. IV. As discussed in [14], this method to compute $\chi(t,t_w)$ is much more efficient than traditional methods where the perturbation is switched on.

In stationary states, a simplified expression for $\chi(t,t_w)$ can be obtained that makes the role of TRI evident. In order to do this, let us consider the integral

$$I(t,t_w) = \frac{1}{2}\int_{t_w}^{t} dt'\langle B_i(t)\sigma_i(t')\rangle. \qquad (31)$$

Enforcing Eq. (23), proceeding as in [14], the integrand can be written as

$$\langle B_i(t)\sigma_i(t')\rangle = \frac{\partial\langle \beta_j(t)\sigma_j(t)\sigma_i(t')\rangle}{\partial t}. \qquad (32)$$

Using Eq. (32), replacing d/dt with $-d/dt'$, due to TTI, and carrying out the integration one has

$$I(t,t_w) = \frac{1}{2}[A(t,t) - \langle \beta_i(t)\sigma_i(t)\sigma_i(t_w)\rangle]. \qquad (33)$$

Adding and subtracting $I(t,t_w)$ on the right-hand side, Eq. (30) can be cast in the form (4), with

$$\Delta(\tau) = \frac{1}{2}\left\{ \langle \beta_i(t)\sigma_i(t)\sigma_i(t_w)\rangle - \langle \beta_i(t_w)\sigma_i(t)\sigma_i(t_w)\rangle \right. \\ \left. + \int_{t_w}^{t} dt'[\langle \sigma_i(t)B_i(t')\rangle - \langle B_i(t)\sigma_i(t')\rangle]\right\}. \qquad (34)$$

If TRI is also obeyed, as in equilibrium states, one has

$$\langle \beta_i(t)\sigma_i(t)\sigma_i(t_w)\rangle = \langle \beta_i(t_w)\sigma_i(t)\sigma_i(t_w)\rangle$$

and

$$\langle \sigma_i(t)B_i(t')\rangle = \langle B_i(t)\sigma_i(t')\rangle,$$

so that $\Delta(\tau) = 0$. Equation (4) becomes a linear relation formally identical to Eq. (5). When TRI does not hold, instead, $\Delta(\tau) \neq 0$ and the relation between $\chi(\tau)$ and $A(\tau)$ is no longer linear. As we will see in Sec. IV A, this is an efficient tool to check if a stationary state is invariant under time reversal and, if not, to quantify TRI violations.

IV. NUMERICAL RESULTS

In this section, we present a numerical investigation of the dynamical properties of the model and, in particular, of the fluctuation-dissipation relation (30). We chose unperturbed transition rates of the Metropolis type for single spin flip on site i

$$w([\sigma] \to [\sigma']) = \min(1, \exp\{-\beta_i[\sigma](H[\sigma'] - H[\sigma])\}), \qquad (35)$$

which, as can be easily checked, obey Eq. (8). Up down symmetry implies that $\beta_i[\sigma]$ does not depend on the sign of the Weiss field, $\beta_i[\sigma] = \beta(|H_i|)$. In the following, we will consider a system on a square lattice in two dimensions. In this case, the only possible values of the local field are $H_i[\sigma]/J = 0, \pm 2, \pm 4$; the model is then fully defined by assigning the three parameters $\beta(0)$, $\beta(2J)$, and $\beta(4J)$. Moreover, with the Hamiltonian (7), the transition rates (35) do not depend on $\beta(0)$. Then, at this level, the couple of inverse temperatures $\beta(2J)$ and $\beta(4J)$ is sufficient to characterize the model. However, $\beta(0)$ becomes relevant if the system is perturbed by an external magnetic field in order to measure response functions. Actually, this quantity enters, through $A(t,t')$ and B_i,
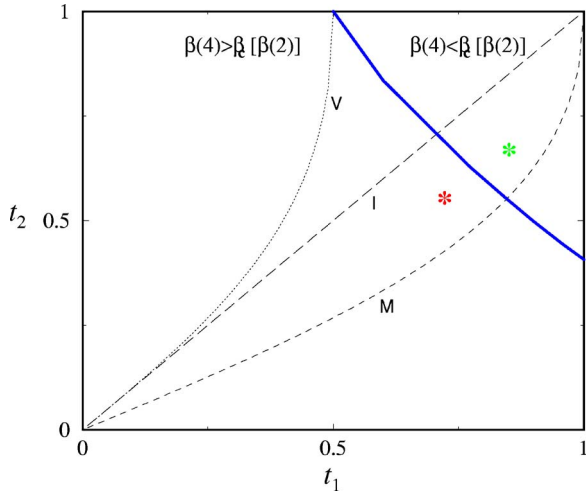
FIG. 1. (Color online) The phase diagram of the IWDB model, where $t_i = \tanh(2\beta_i)$. The three broken curves correspond, respectively, V to the Voter model, I to KIM, and M to the extreme or majority model. The two asterisks correspond to the two sets of parameters used in simulations.

the expressions (26) and (29) of the response functions. Then, response functions depend on $\beta(0)$, as already found numerically in Ref. [13].

The *phase diagram* of the IWDB was studied in Refs. [7,8]. It was shown that in the plane of the parameters $\beta(2)$ and $\beta(4)$, one can identify two regions separated by a *critical line* $\beta(4) = \beta_c[\beta(2)]$, as shown in Fig. 1. The critical line starts at the inverse temperatures $\beta(2) = \frac{1}{2}$ arctanh $\frac{1}{2}$, $\beta(4) = \infty$, corresponding to the Voter model, where the transition occurs in the absence of bulk noise, passes through the Onsager critical point with $\beta(2) = \beta(4) = \frac{1}{2}$ arcsinh(1) and ends at $\beta(2) = \infty$, $\beta(4) \simeq 0.22$, corresponding to the extreme model, where the transition occurs in the absence of interfacial noise. For $\beta(4) < \beta_c[\beta(2)]$ one has a high-temperature phase similar to the paramagnetic phase of the KIM. Starting from any initial state, the system quickly attains a stationary state where the magnetization

$$m = \langle \sigma_i \rangle \quad (36)$$

vanishes. For $\beta(4) > \beta_c[\beta(2)]$, there is a low-temperature phase similar to a ferromagnetic phase. Here there are two possible dynamical situations, depending on whether the system enters a state with broken symmetry, namely, with $m = \pm M_{BS} \neq 0$, or not. In the former case, a stationary state is entered; in the latter there is a phase-ordering process and the system ages. We will consider these cases separately in Secs. IV A and IV B, where we will present the results of numerical simulations of a two-dimensional system on a square lattice of size 1000^2, with $J=1$.

A. Stationary states

1. $\beta(4) < \beta_c[\beta(2)]$

We have prepared the system in the stationary state at the inverse temperatures $\beta(0) = 0.80$, $\beta(2) = 0.44$, and $\beta(4)$
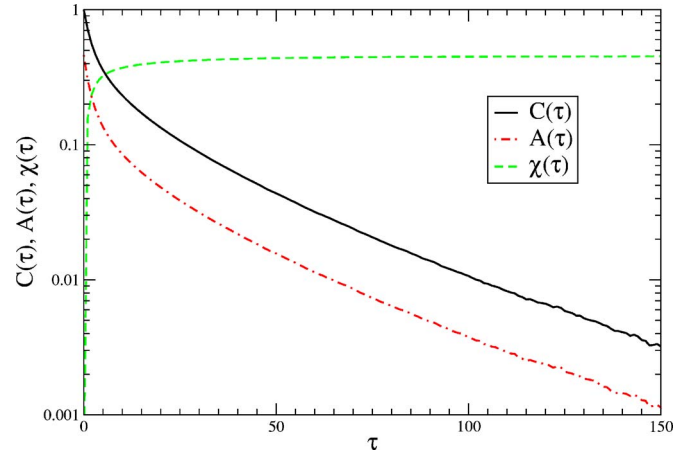

FIG. 2. (Color online) $C(\tau)$, $A(\tau)$, and $\chi(\tau)$ are plotted against τ.

$= 0.30$, which correspond to the paramagnetic phase. This state is quickly entered by the system by letting it evolve from any initial condition. In the stationary state, we checked that $m=0$ and that two-time quantities are functions of the time difference τ alone. In the following, time will be measured in Monte Carlo steps. $C(\tau)$, $A(\tau)$, and $\chi(\tau)$ are shown in Fig. 2.

The behavior of $C(\tau)$ is analogous to that observed in the KIM. Starting from $C(0) = \langle \sigma_i^2(t_w) \rangle = 1$, the correlation function exponentially decays to zero. This is due to the decorrelation of the spin for large time differences $\lim_{\tau \to \infty} C(\tau) = \lim_{\tau \to \infty} \langle \sigma_i(t_w + \tau) \sigma_i(t_w) \rangle = \langle \sigma \rangle \langle \sigma \rangle = m^2 = 0$. Here we have introduced the simplified notation $\langle \sigma \rangle = \langle \sigma_i(t) \rangle$ to indicate that $\langle \sigma_i(t) \rangle$ does not depend on time t nor on site i due to TTI and space homogeneity. We will use this notation also in the following, dropping time and/or space variables whenever ensemble averages do not depend on them.

$A(\tau)$ behaves similarly. From the definition (27), its equal time value is the average inverse temperature of the bath, $A(0) = \langle \beta \rangle = 0.46$. For large times difference also, this correlation function decays to zero, since $\lim_{\tau \to \infty} A(\tau) = m \langle \sigma \beta \rangle = 0$. Note that $C(\tau)$ and $A(\tau)$ are proportional for large τ but not for small τ. This fact will be relevant in the following, when discussing the possibility to define a thermodynamic temperature from the parametric plots $\hat{\chi}(C)$ and $\hat{\chi}(A)$ obtained plotting $\chi(\tau)$ vs $C(\tau)$ or $A(\tau)$, respectively.

The behavior of $\chi(\tau)$ is also closely related to what is known for the KIM. This quantity starts from $\chi(0) = 0$ and saturates exponentially to a constant value χ_∞. In the KIM, this value is the equilibrium susceptibility, namely, the inverse temperature, $\chi_\infty = \beta$. One could conjecture that in the IWBD this result can be generalized to $\chi_\infty = \langle \beta \rangle$. However, as we will see shortly, this is not true due to lack of TRI. In order to discuss this point let us consider, in Fig. 3, the parametric plot of $\hat{\chi}(A)$.

For the largest values of A, a relation formally identical to Eq. (5) is obeyed. Recalling Eq. (4), this implies that the term $\Delta(\tau)$ is negligible. This, in turn, shows that TRI is satisfied in this time domain. This is reminiscent of what happens in out of equilibrium systems in contact with a single
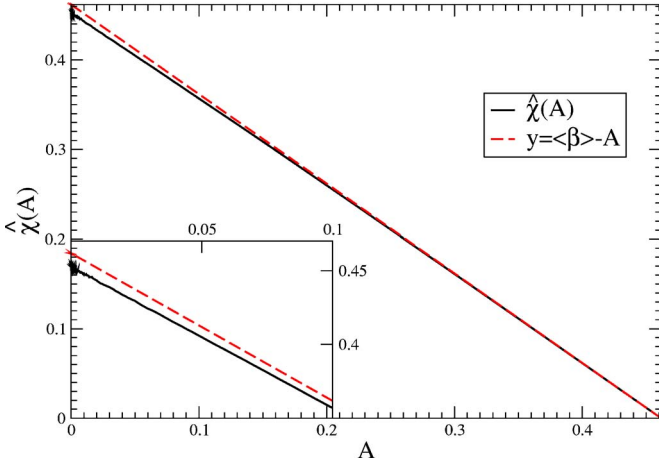
FIG. 3. (Color online) The parametric plot $\chi(A)$. In the inset, the small A sector is magnified.

FIG. 4. (Color online) The parametric plot $\chi(C)$.

reservoir, where the linear FDT relation (5) is found on the right-hand side of the $\hat{\chi}(A)$ parametric plot, despite the system is not in equilibrium. Since in this case $A(0)=\langle\beta\rangle$, Eq. (5) can be written as

$$\hat{\chi}(A) = \langle\beta\rangle - A. \tag{37}$$

This shows that the average bath temperature $\langle\beta\rangle$ enters the relation between $\chi(\tau)$ and $A(\tau)$, as a natural generalization of what happens in the KIM where one has $\chi(A)=\beta-A$. This behavior can be explained recalling that, in this sector of the plot, namely, for $\tau\simeq 0$, the response is provided by the fastest dynamical features. These are the microscopic flipping of single spins that locally and instantaneously equilibrate at the current inverse temperature $\beta_i[\sigma]$. Since the system is translationally invariant, by taking ensemble averages one gets a sort of FDT with respect to the average bath temperature $\langle\beta\rangle$, namely, Eq. (37). As larger values of τ are considered, corresponding to lower values of $A(\tau)$, slower dynamical features are probed that cannot follow the variations of $\beta_i[\sigma]$. In this sector, the contribution of $\Delta(\tau)$ becomes relevant and the parametric plot $\chi(A)$ deviates from the straight line. Recalling the discussion of Sec. III, $\Delta(\tau)\neq 0$ is related to the lack of TRI. These considerations then suggest that the parametric plot can be used as a convenient tool to detect and quantify TRI violations in nonequilibrium stationary states. Breakdown of TRI is also responsible for the saturation of $\chi(\tau)$ to a value $\chi_\infty < \langle\beta\rangle$.

The calculations of Sec. III clearly show that $A(t,t_w)$ is the correlation function naturally associated to $\chi(t,t_w)$, rather than the autocorrelation function $C(t,t_w)$, which does not enter the generalization of the fluctuation-dissipation theorem (30). Nevertheless, we also consider, in Fig. 4, the parametric plot of $\chi(\tau)$ vs $C(\tau)$, since, by analogy with systems with detailed balance, this relation is often considered in the literature [13].

Also in this case, for the larger values of C, the curve $\chi(C)$ obeys a linear relation $\chi(C)=\hat{\beta}(1-C)$, with $\hat{\beta}=0.55$. Although this fact has suggested the interpretation [13] of
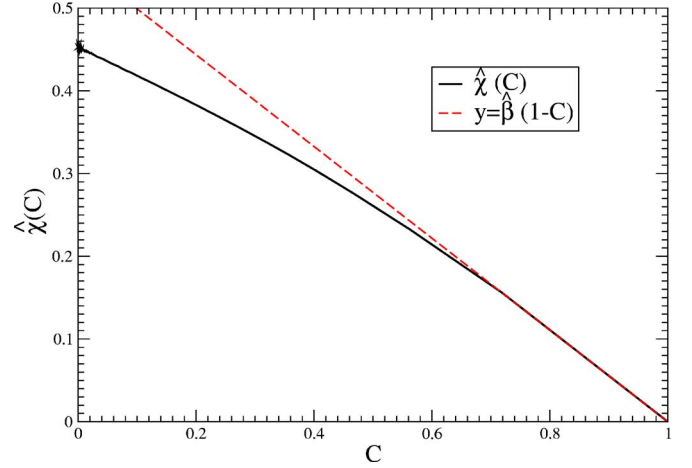
$\hat{\beta}^{-1}$ as a thermodynamic temperature, since $C(t,t_w)$ does not enter the fluctuation-dissipation relation (30), this reading is unmotivated. Note, in fact, that $\hat{\beta}\neq\langle\beta\rangle$. Indeed, one should have $\hat{\beta}=\langle\beta\rangle$ if $C(\tau)\propto A(\tau)$ in the regime considered, namely, for small τ. Instead, as discussed previously, this is not the case. Actually, we have checked that $\hat{\beta}\neq\langle\beta\rangle$ even when the choice of $\beta(0)$ proposed in [13] is made, both in the high- and low-temperature phases.

For small values of C, the curve $\chi(C)$ strongly deviates from the straight line. The parametric plot $\chi(C)$ can be compared to that found in stationary states of other systems without detailed balance and, in particular, in an exactly soluble model of binary systems under shear flow [10], where a similar pattern was found.

2. $\beta(4) > \beta_c[\beta(2)]$

We have prepared the system in the stationary state at the inverse temperatures $\beta(0)=0.80$, $\beta(2)=0.68$, and $\beta(4)=0.37$, corresponding to the ferromagnetic phase. This state is quickly entered by the system by letting it evolve from any initial condition with a broken symmetry $m\neq 0$. We consider the stationary state where the magnetization attains the positive value $m=M_{BS}=0.83$. Two time quantities, denoted by $C_{BS}(\tau)$, $A_{BS}(\tau)$, and $\chi_{BS}(\tau)$ are plotted in Fig. 5.

$C_{BS}(\tau)$ behaves similarly to the KIM. It decays from $C_{BS}(0)=1$ to the large τ value $C_{BS}(\infty)=\langle\sigma\rangle\langle\sigma\rangle=M_{BS}^2=0.69$. Analogously, $A_{BS}(\tau)$ decays from $A_{BS}(0)=\langle\beta\rangle=0.45$, to $A_{BS}(\infty)=\langle\sigma\rangle\langle\sigma\beta\rangle=M_{BS}\langle\sigma\beta\rangle=0.30$. $\chi_{BS}(\tau)$ grows from zero up to the constant value χ_∞. If TRI were obeyed, from Eq. (4) one should have $\chi_\infty=\langle\beta\rangle-M_{BS}\langle\sigma\beta\rangle$. Indeed, for the KIM, this equation gives $\chi_\infty=\beta(1-M_{BS}^2)$, which is, in fact, the equilibrium susceptibility. However, TRI is violated in the IWDB, as it is clear from the fact that the parametric plot of $\chi_{BS}(\tau)$ vs $A_{BS}(\tau)$, shown in Fig. 6, is not a straight line.

Deviations from the straight line are due to the term $\Delta(\tau)$ in Eq. (4), which signals the breakdown of TRI, and makes $\chi_{BS}(\tau)$ saturate to a value $\chi_\infty < \langle\beta\rangle - M_{BS}\langle\sigma\beta\rangle$. Note, however, that also in this case for the largest values of A_{BS}, the linear relation $\chi_{BS}(A_{BS})=\langle\beta\rangle-A_{BS}$ is obeyed, as in the para-
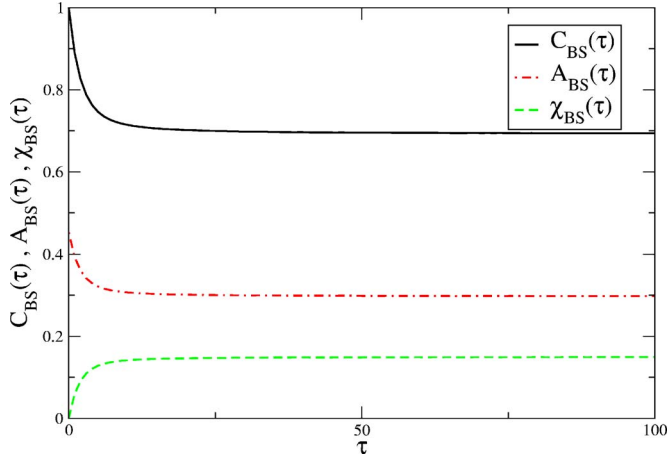
FIG. 5. (Color online) $C(\tau)$, $A(\tau)$, and $\chi(\tau)$ are plotted against τ.

FIG. 7. (Color online) The parametric plot $\chi_{BS}(C_{BS})$.

magnetic phase, implying that TRI is satisfied in this time domain and that the average temperature $\langle\beta\rangle$ can be extracted from this region of the plot, in analogy with the KIM.

In Fig. 7, the parametric plot of $\chi_{BS}(\tau)$ versus $C_{BS}(\tau)$ is also shown. This plot is analogous to the one found in the paramagnetic phase, and similar considerations can be made. In particular, we find $\chi_{BS}(C_{BS})=\hat{\beta}\cdot(1-C_{BS})$, with $\hat{\beta}=0.54$, for the largest values of C. As already discussed in Sec. IV A 1, there is no reason to interpret this quantity as an inverse temperature, and again $\hat{\beta}\neq\langle\beta\rangle$.

Let us also introduce the connected two-time quantities that will be used in Sec. IV B. Using the general definition of the connected correlation function $\mathcal{D}$ between two observables O and O', $\mathcal{D}=\langle OO'\rangle-\langle O\rangle\langle O'\rangle$, the connected two-time quantities associated to $C_{BS}(\tau)$ and $A_{BS}(\tau)$ are

$$\mathcal{C}_{BS}(\tau) = C_{BS}(\tau) - M_{BS}^2, \tag{38}$$

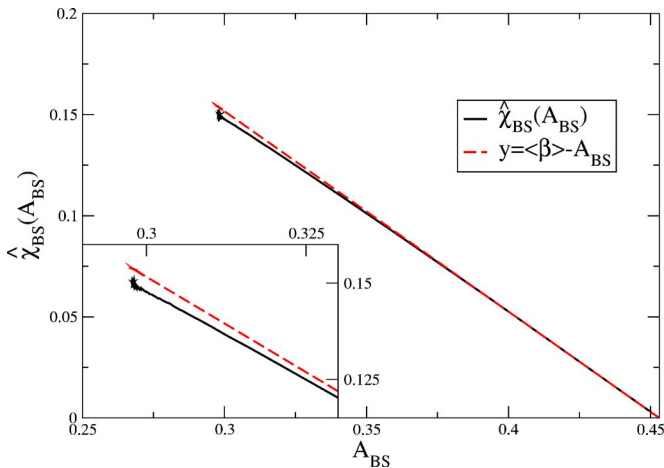$$\mathcal{A}_{BS}(\tau) = A_{BS}(\tau) - M_{BS}\langle\sigma\beta\rangle. \tag{39}$$

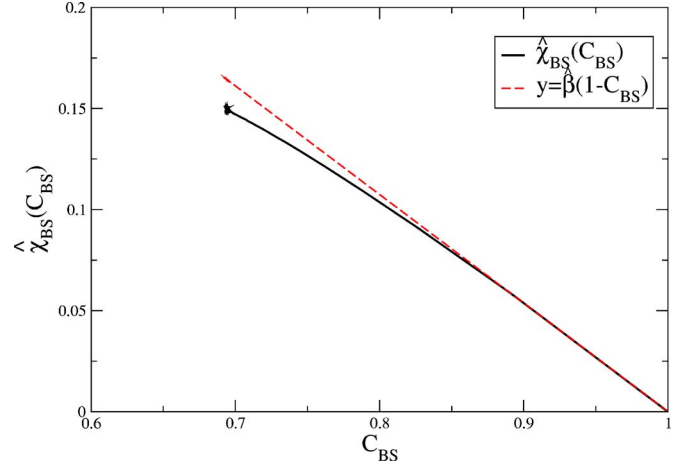FIG. 6. (Color online) The parametric plot $\chi_{BS}(A_{BS})$. In the inset, the small A_{BS} sector is magnified.

B. Aging dynamics

In this section, we study the nonequilibrium process following a quench from an initial disordered state with $m=0$ to the final inverse temperatures $\beta(0)=0.80$, $\beta(2)=0.68$, and $\beta(4)=0.37$. Note that these are the same temperatures of Sec. IV A 2, corresponding to a point in the ordered phase. In this case, one observes a phase-ordering process where domains of two phases with $m=\pm M_{BS}$ coarsen [8]. In the interior of such domains, the system is found in the stationary state studied in Sec. IV A. In analogy to what is known for the KIM, and, more generally, in aging systems [11], we expect quantities such as the equal time correlation function

$$G(r,t) = \langle\sigma_i(t)\sigma_j(t)\rangle, \tag{40}$$

i and j being two sites whose distance is r, or two-time correlation functions and response to take the additive structure [12]

$$G(r,t) = G_{st}(r) + G_{ag}(r,t), \tag{41}$$

$$C(t,t_w) = C_{st}(\tau) + C_{ag}(t,t_w), \tag{42}$$

$$A(t,t_w) = A_{st}(\tau) + A_{ag}(t,t_w), \tag{43}$$

$$\chi(t,t_w) = \chi_{st}(\tau) + \chi_{ag}(t,t_w). \tag{44}$$

The presence of the stationary state in the bulk of the growing domains is the origin of the contributions $G_{st}(r)$, $C_{st}(\tau)$, $A_{st}(\tau)$, and $\chi_{st}(\tau)$, while the terms $G_{ag}(r,t)$, $C_{ag}(t,t_w)$, $A_{ag}(t,t_w)$, and $\chi_{ag}(t,t_w)$ take into account the aging degrees of freedom in the system.

In the KIM quenched to the final temperature β^{-1}, $G_{st}(r)$ is the correlation function of the stationary state with broken symmetry at the same temperature β^{-1}, namely, the correlation $G_{BS}(r)$. We define $G_{st}(r)$ in complete analogy for the IWDB, $G_{BS}(r)$ being the quantity measured in the stationary state at the same inverse temperatures. $G_{ag}(r,t)$ can then be obtained by subtraction, by using Eq. (41). In the scaling regime, $G_{ag}(r,t)$ obeys
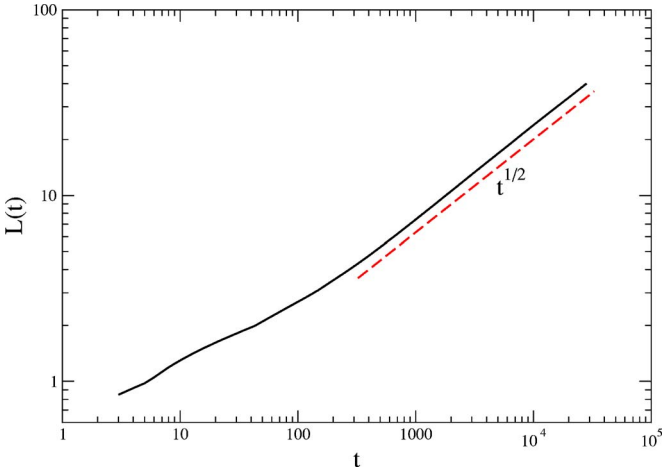
FIG. 8. (Color online) The typical size $L(t)$ of domains.

$$G_{ag}(r,t) = M_{BS}^2 g(x), \qquad (45)$$

$x = \frac{r}{L(t)}$. This property will be tested below.

The typical size of domains can then be computed as the half height width of $G_{ag}(r,t)$. This quantity is shown in Fig. 8. After the initial transient, $L(t)$ has a power-law behavior $L(t) \sim t^{1/z}$. We measure $1/z = 0.50$, as for the KIM.

Coming back to the scaling (45), in order to check this form we plot, in Fig. 9, $G_{ag}(r,t)/M_{BS}^2$ against x for different values of t. According to Eq. (45) one should find data collapse for different times. Actually, the collapse is good even if worse than in the KIM, particularly for $x \simeq 0$. Note also that the form of the scaling function $g(x)$ is very similar to that of the KIM.

Let us now discuss two-time quantities. Analogously to what was discussed above for $G(r,t)$, in the KIM one has

$$C_{st}(\tau) = \mathcal{C}_{BS}(\tau). \qquad (46)$$

We define $C_{st}(\tau)$ in complete analogy for the IWDB, and the same is assumed for $A_{st}(\tau)$,
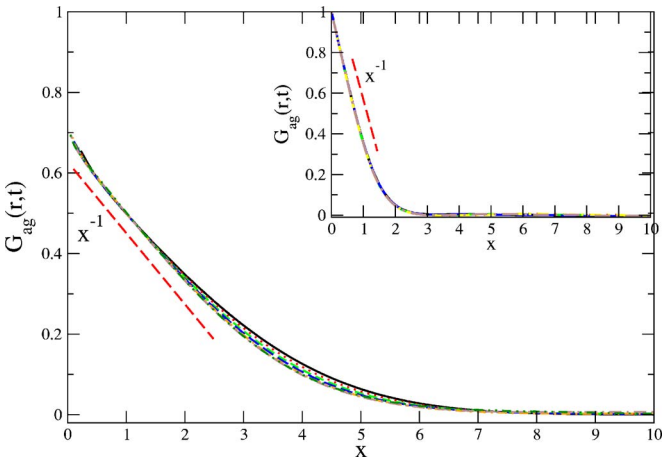

FIG. 9. (Color online) $G_{ag}(r,t)$ is plotted against x for different times t generated from $t_n = \text{int}[\exp(n/2)]$ with n ranging from 14 to 20. In the inset $G_{ag}(r,t)$ for the same values of t is plotted for the KIM quenched at $T = 0$. One observes the same small x behavior $g(x) \sim 1/x$.

$$A_{st}(\tau) = \mathcal{A}_{BS}(\tau). \qquad (47)$$

For the integrated autoresponse function, in systems with a constant β, $\chi_{st}(\tau)$ is the response produced in the bulk of domains and is defined as the quantity related by Eq. (4) to the stationary parts of the correlation functions. In this case $\Delta(\tau) = 0$, Eq. (4) is the FDT (5) and

$$\chi_{st}(\tau) = A_{st}(0) - A_{st}(\tau) = \mathcal{A}_{BS}(0) - \mathcal{A}_{BS}(\tau) = A_{BS}(0) - A_{BS}(\tau). \qquad (48)$$

Clearly, since $\chi_{st}(\tau)$ is related by FDT (5) to the correlation function of the broken symmetry equilibrium state, it is the integrated autoresponse function of that state. Then one has

$$\chi_{st}(\tau) = \chi_{BS}(\tau). \qquad (49)$$

For the present model, in full analogy to the case with constant β, we use Eq. (49) to define $\chi_{st}(\tau)$. Namely, $\chi_{st}(\tau)$ is the quantity measured in the stationary state at the same inverse temperatures in the previous section. Let us now turn to discuss the properties of the aging contributions in Eqs. (42)–(44). Recalling the behavior of $C_{BS}(\tau)$ one concludes that $C_{ag}(t,t_w)$ decays from $C_{ag}(t_w,t_w) = M_{BS}^2$ to zero, as in the KIM. In analogy to the KIM, we expect it to obey the scaling form

$$C_{ag}(t,t_w) = h_C(y), \qquad (50)$$

where $y = t/t_w$, with the power law $h_C(y) \sim y^{-\lambda/z}$ for large y, λ being the Fisher-Huse exponent. Analogously, given the behavior of $A_{BS}(\tau)$ discussed in the previous section, one concludes that $A_{ag}(t,t_w)$ decays from $A_{ag}(t_w,t_w) = M_{BS}\langle \sigma\beta \rangle = 0.30$ to zero. We expect a scaling form

$$A_{ag}(t,t_w) = h_A(y), \qquad (51)$$

as for $C_{ag}(t,t_w)$. The response $\chi_{ag}(t,t_w)$ is produced by the interface degrees of freedom whose number goes to zero during the ordering process. For this reason, in the $d=2$ KIM this quantity after reaching a maximum for $y \sim 1$ decays to zero. A scaling behavior is obeyed, namely,

$$\chi_{ag}(t,t_w) = t_w^{-a_\chi} f(y), \qquad (52)$$

with the power law $f(y) \sim y^{-a_\chi}$ for large y and a_χ consistent with $a_\chi = \frac{1}{4}$ [17]. In order to check these scaling forms, we have extracted the aging terms as

$$C_{ag}(t,t_w) = C(t,t_w) - C_{st}(\tau) = C(t,t_w) - \mathcal{C}_{BS}(\tau), \qquad (53)$$

$$A_{ag}(t,t_w) = A(t,t_w) - A_{st}(\tau) = A(t,t_w) - \mathcal{A}_{BS}(\tau), \qquad (54)$$

and

$$\chi_{ag}(t,t_w) = \chi(t,t_w) - \chi_{st}(\tau) = \chi(t,t_w) - \chi_{BS}(\tau). \qquad (55)$$

According to Eq. (50), curves $C_{ag}(t,t_w)$ corresponding to different values of t_w should collapse when plotted against y. This type of plot is shown in Fig. 10, which shows that the data collapse is not very good for all times. This can be associated to the presence of preasymptotic effects. However, the quality of the collapse gets better for the largest values of t_w and t. Indeed, while the two curves with the
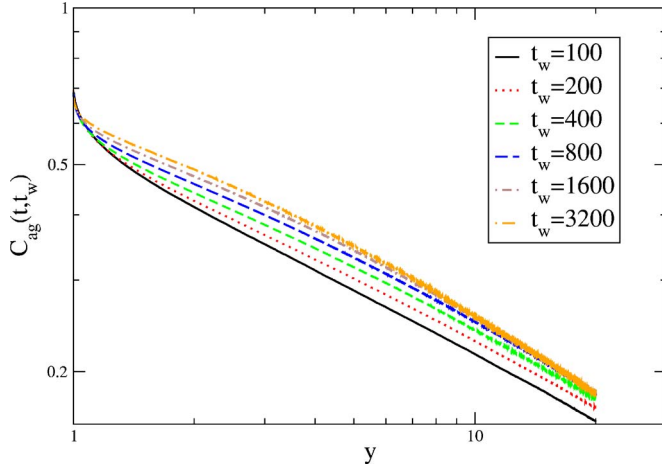
FIG. 10. (Color online) $C_{ag}(t,t_w)$ is plotted against y.

smallest values of t_w ($t_w=100,200$) do not collapse at all, there is a tendency to a better collapse as t_w gets larger. For the two largest values ($t_w=1600,3200$) one has a nice collapse from $y\simeq5$ onward.

A similar situation is observed for the correlation $A_{ag}(t,t_w)$, as shown in Fig. 11. Similar to what is found in stationary states, we find that $C_{ag}(t,t_w)\propto A_{ag}(t,t_w)$ is largest for y.

According to Eq. (52), the exponent a_χ can be extracted as the slope of the double-logarithmic plot of $\chi_{ag}(t,t_w)$ against t_w, by keeping y fixed. We do this in Fig. 12, for different choices of y. We observe a good power-law behavior, for every value of y. Best fit exponents are in the range 0.23–0.28, depending on y, suggesting that the same value $a_\chi=\frac{1}{4}$ of the KIM is found here. Then, in order to check the data collapse, in Fig. 13 we plot $t_w^{a_\chi}\chi_{ag}(t,t_w)$, with $a_\chi=\frac{1}{4}$, against y, for different t_w. The collapse is, indeed, rather good for the two largest values of t_w, implying that Eq. (52) with an exponent consistent with $a_\chi=\frac{1}{4}$, as in the KIM, is asymptotically obeyed. This result complements those of Refs. [7,8], where it was shown that the IWDB has the same equilibrium critical exponent and the same persistence exponent of the KIM. This strongly indicates that this two model
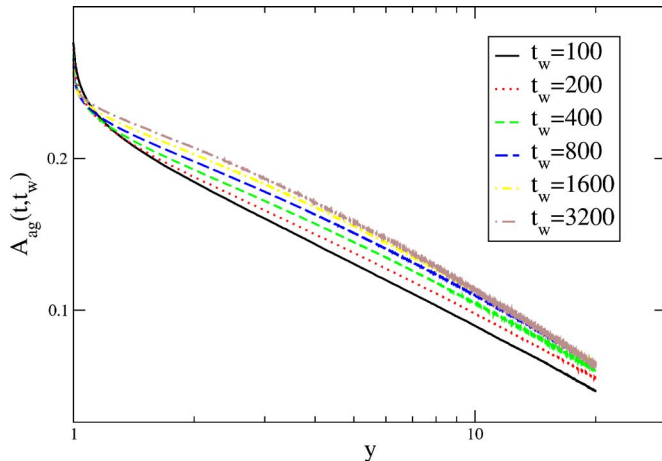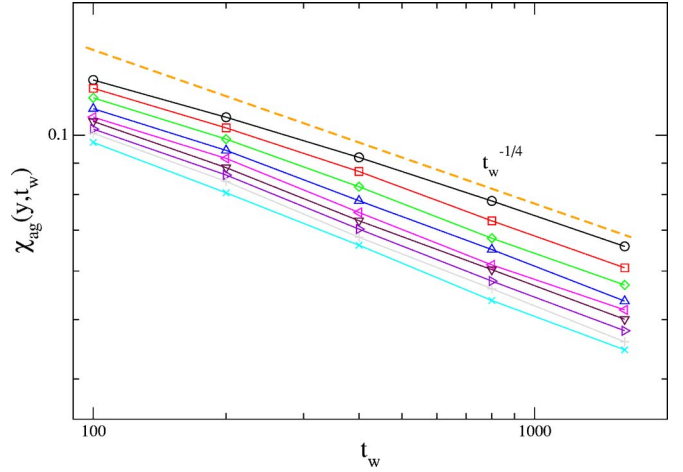

FIG. 12. (Color online) $\chi_{ag}(t,t_w)$ is plotted against t_w for fixed $y=t/t_w$, and $y=3,5,7,9,11,13,15,17,19$ from top to bottom. The dashed line is the power law $t_w^{-(1/4)}$.

belongs to the same *equilibrium* and *nonequilibrium* universality class.

In Fig. 14, the parametric plot, of $\hat{\chi}(A)$ is shown. In order to understand this plot, one has to consider separately the short time separation regime (ST), namely, the limit $t_w\to\infty$ with $\tau/t_w\ll1$, and the large time separation regime (LT), where $t_w\to\infty$ with $\tau/t_w\gg1$. In the ST, given the scalings (50)–(52), the aging parts of two time functions remain equal to their equal time value. For $A_{ag}(t,t_w)$, one has $A_{ag}(t,t_w)=A_{ag}(t_w,t_w)=M_{BS}\langle\sigma\beta\rangle$. Then, from Eq. (43) one has $A(t,t_w)=M_{BS}\langle\sigma\beta\rangle+A_{st}(\tau)=M_{BS}\langle\sigma\beta\rangle+\mathcal{A}_{BS}(\tau)$. Recalling the behavior of $\mathcal{A}_{BS}(t,t_w)$ one concludes that, in the ST, $A(t,t_w)$ decays from $\langle\beta\rangle=0.45$ to $M_{BS}\langle\sigma\beta\rangle=0.3$. In this time domain, one has $\chi_{ag}(t,t_w)=0$ and, hence, $\chi(t,t_w)=\chi_{st}(\tau)=\chi_{BS}(\tau)$. Therefore, on the right-hand side of the parametric plot of Fig. 14, for $A\gtrsim0.3$, one should find exactly the same curve found in the stationary broken symmetry state, namely, Fig. 6. This curve is the broken line in Fig. 14. This implies that for the largest values of A, let us say for $A>0.4$, Eqs. (5) and (37) are obeyed, as discussed in Sec. IV A 2.

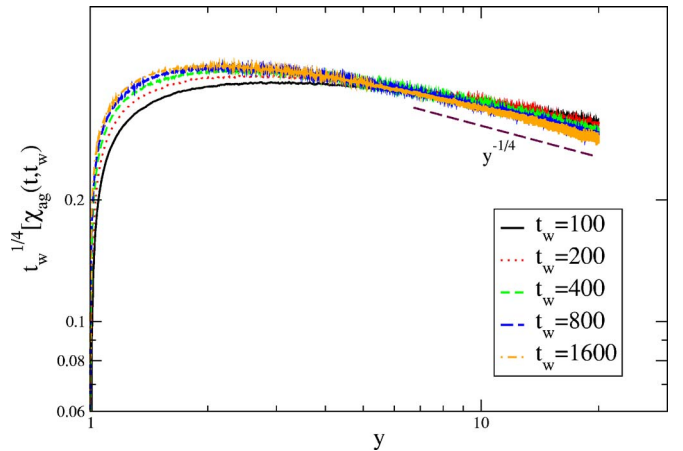FIG. 11. (Color online) $A_{ag}(t,t_w)$ is plotted against $y=t/t_w$.

FIG. 13. (Color online) $t_w^{1/4}\chi_{ag}(t,t_w)$ is plotted against $y=t/t_w$. The dashed line is the power law $y^{-(1/4)}$.

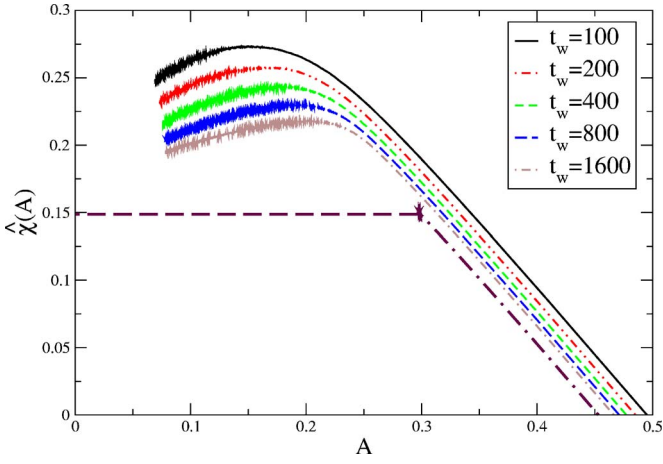FIG. 14. (Color online) The parametric plot $\chi(A)$. The dashed line is the expected asymptotic behavior, for $t_w=\infty$.

FIG. 15. (Color online) The parametric plot $\chi(C)$. The dashed line is the expected asymptotic behavior for $t_w=\infty$.

The numerical simulations can only access finite t_w, and some deviations from the asymptotic curve are then observed. Note, in particular, that the equal time value $A(t_w,t_w)$ has a weak time dependence. In this case there is a monotonous decrease of this quantity. Recalling that $A(t_w,t_w)=\langle\beta(t_w)\rangle$, this means that in the kinetic process the average temperature is slightly increasing. This happens because in the phase-ordering process the fraction of bulk spins, with $\beta_i=\beta(4)$, grows in time. Since in this case $\beta(4)<\beta(2)<\beta(0)$, this corresponds to an increase of the average temperature. Despite these finite time effects, however, the data clearly show that the curves converge to the broken line increasing t_w.

On the left-hand side of the plot, the LT is probed. In this regime, $\mathcal{A}_{BS}(t,t_w)=0$ and $A(t,t_w)=A_{ag}(t,t_w)$ decays from $M_{BS}\langle\sigma\beta\rangle$ to zero. On the other hand, $\chi_{st}(\tau)$ has already reached its asymptotic value $\chi_{st}(\tau)=\chi_\infty$ so that $\chi(t,t_w)=\chi_\infty+\chi_{ag}(t,t_w)$. According to the scaling form (52), $\chi_{ag}(t,t_w)$ vanishes in the large t_w limit. Then, for $t_w=\infty$ on the left-hand side of Fig. 14, one should find the horizontal straight line typical of phase-ordering systems. For finite values of t_w, $\chi_{ag}(t,t_w)$ still contributes to the response and the curve overshoots the asymptotic value χ_∞. However, as shown in Fig. 14, the asymptotic curve is approached increasing t_w.

Let us now consider the parametric plot of $\chi(t,t_w)$ versus $C(t,t_w)$, shown in Fig. 15. Repeating the same considerations as for the previous figure, one concludes that on the right-hand side of Fig. 15, the curves approach the curve of $\chi_{BS}(\tau)$ against $C_{BS}(\tau)$ of Sec. IV A 2, namely, Fig. 7, in the large t_w limit. We stress again that, as already discussed in Sec. IV A, there is no reason to associate the quantity $\hat{\beta}^{-1}$ extracted from this sector of the plot, to a thermodynamic temperature, as claimed in [13].

On the left-hand side of Fig. 15, our data are consistent with a convergence to the flat line $\chi(C)=\chi_\infty$, typical of phase ordering. Note, however, that the whole shape of the parametric plot is different from that of the KIM, due to the different relation between the stationary parts of $\chi(t,t_w)$ and $C(t,t_w)$, which shows up in the large-C region. Moreover, also in this case the plot is similar to what observed in binary systems under shear [10].
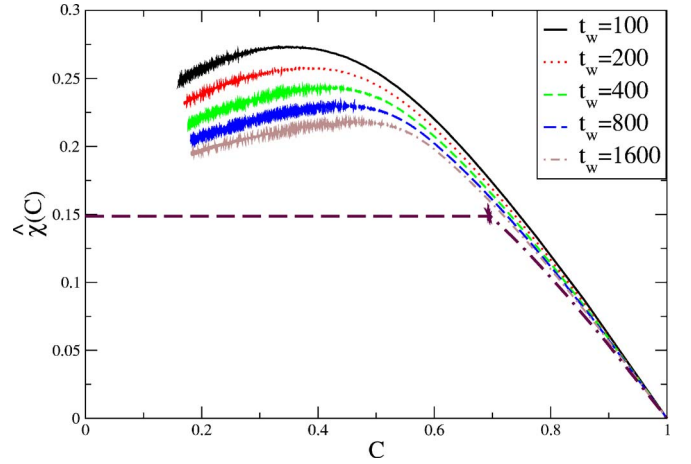
V. SUMMARY AND CONCLUSIONS

In this paper we have studied a modified Ising model, the IWDB, where the temperature entering the transition rates depends on space and time through the system configuration and detailed balance is violated. This model is known to share many properties of the Ising model, including the phase diagram, critical exponents [7], and nonstationary dynamics [8].

In systems with detailed balance a relation between the integrated response function $\chi(t,t_w)$, the autocorrelation function $C(t,t_w)$ and the asymmetry $\Delta(t,t_w)$, a term related to the possible lack of TRI, can be obtained under general assumptions [9,14]. This fluctuation-dissipation relation applies also in nonstationary states, namely, out of equilibrium.

In this paper we have derived an analogous fluctuation-dissipation relation for the IWDB. The result is similar to the case with detailed balance, but the role played by $C(t,t_w)$ is now played by the correlation $A(t,t_w)$ between the spins and the time-dependent local inverse temperature. Since $\beta_i(t_w)$ enters the transition rates and is, therefore, correlated to the spin configuration, $A(t,t_w)$ is not simply related to $C(t,t_w)$. It is therefore natural to consider the relation $\hat{\chi}(A)$ for which a generalization of what is known in systems with detailed balance seems to be possible, instead of the relation $\hat{\chi}(C)$ whose meaning, as far as we can see, remains unclear.

In the stationary states of the model, which are the counterparts of the Ising equilibrium states, the fluctuation-dissipation relation (4) is formally similar to the FDT (5), with the important difference of a nonvanishing $\Delta(t,t_w)$, determined by the violation of TRI. This term makes $\hat{\chi}(A)$ nonlinear. However, for small time differences, namely, for the largest values of A, the asymmetry can be neglected and one recovers a linear relation, as in equilibrium systems, with the average inverse temperature $\langle\beta\rangle$ playing the role of β in equilibrium systems.

After quenching the systems into the ferromagnetic phase, a nonstationary process is observed, similar to the phase-ordering kinetics of the KIM. We find that the response function exponent takes a value consistent with $a_\chi=\frac{1}{4}$, as for the

KIM. This fact complements previous results [7,8] on the universality between the two models, both in *equilibrium* and *out of equilibrium*. The shape of plots $\hat{\chi}(A)$ and $\hat{\chi}(C)$ can also be discussed in strict analogy to what is observed in the KIM. In particular, one finds the flat horizontal line typical of phase-ordering systems. Interestingly, the parametric plot $\hat{\chi}(C)$ is similar to that of a soluble model of sheared binary systems where detailed balance is also violated, suggesting some qualitative similarity between these two model.

[1] We cast a ˆ over the response function to indicate its dependence on the correlation function.

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